HOMEOMORPHIC GRAPHS AND THE THEOREM OF KURATOWSKI

The Polish mathematician Kazimierz Kuratowski discovered an interesting property of planar and non-planar graphs. In fact, it is in honour of Kuratowski that the complete graphs are labelled $K_n$ and the complete bipartite graphs are labelled $K_{m,n}$. However, before we can discuss Kuratowski’s work, we need to look at homeomorphism.

Two graphs $G$ and $H$ are **homeomorphic** if they can be made isomorphic by inserting new vertices (of degree 2) into their existing edges.

For example, the two graphs below are homeomorphic since the addition of the coloured vertices shown will make them isomorphic.

Similarly, all of these graphs are homeomorphic:

Now $K_5$ and $K_{3,3}$ are both non-planar. It is therefore clear that if either $K_5$ or $K_{3,3}$ is a subgraph of a graph $G$, then $G$ must also be non-planar. However, Kuratowski extended this to say that every non-planar graph has a subgraph that is homeomorphic to either $K_5$ or $K_{3,3}$.

Formally we can state Kuratowski’s theorem:

A graph is planar if and only if it contains no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

For example, these diagrams show a non-planar graph and a subgraph homeomorphic to $K_{3,3}$.

A further result is that:

A graph is planar if and only if it contains no subgraph contractible to $K_5$ or $K_{3,3}$ by removing edges from the subgraph and merging the adjacent vertices into one.

For example, we can contract the Peterson graph as shown below, thus proving the Peterson graph is non-planar.
EXERCISE

1. Show that these graphs are homeomorphic:

2. Show that all circuit graphs are homeomorphic to $C_3$.

3. Show that $K_4$ is homeomorphic to $K_{2,2}$.

4. Suppose $G_1$ has $v_1$ vertices and $e_1$ edges and that $G_2$ has $v_2$ vertices and $e_2$ edges and that $G_1$ is homeomorphic to $G_2$. Show that $e_1 - v_1 = e_2 - v_2$.

5. If $G$ is Eulerian and $H$ is homeomorphic to $G$, is $H$ Eulerian?

6. If $G$ is Hamiltonian and $H$ is homeomorphic to $G$, is $H$ Hamiltonian?

7. Use Kuratowski’s theorem to show that $K_n$ is non-planar for $n \geq 5$.

8. Use Kuratowski’s theorem to show that the graphs below are non-planar.

9. Can you use Kuratowski’s theorem to show that the graphs below are non-planar?
HOMEOMORPHIC GRAPHS AND THE THEOREM OF KURATOWSKI - ANSWERS

1. If we add the vertices shown, the resulting graphs are isomorphic.

2. Consider the general circuit graph with \( n \) vertices, i.e., \( C_n \) where \( n \geq 3 \).
   
   If we add a vertex of degree 2 into any existing edge, we generate the circuit graph with \( n + 1 \) vertices, i.e., \( C_{n+1} \).
   
   \( \therefore \) \( C_n \) is homeomorphic to \( C_{n+1} \) for all \( n \geq 3 \).
   
   \( \therefore \) by induction, all circuit graphs are homeomorphic to \( C_3 \).

3. Given \( K_3 \), we can add the vertex shown:

   The graph is now isomorphic to \( K_{2,2} \):

   Hence \( K_3 \) and \( K_{2,2} \) are homeomorphic.

4. If \( G_1 \) and \( G_2 \) are homeomorphic, then we can add vertices of degree 2 into their existing edges in some manner so as to form isomorphic graphs \( H_1 \) and \( H_2 \).

   We suppose \( H_1 \) and \( H_2 \) each have \( v \) vertices.

   So, to form \( H_1 \) from \( G_1 \), we add \( (v - v_1) \) vertices of degree 2. The sum of the degrees of the vertices of \( G_1 \) is \( 2v_1 \), so the sum of the degrees of the vertices of \( H_1 \) is \( 2v_1 + 2(v - v_1) \).

   Similarly, the sum of the degrees of the vertices of \( H_2 \) is \( 2v_2 + 2(v - v_2) \).

   But \( H_1 \) and \( H_2 \) are homeomorphic, so

   \[ 2v_1 + 2(v - v_1) = 2v_2 + 2(v - v_2) \]

   \( \therefore \) \( v_1 - v_1 = v_2 - v_2 \) as required.

5. If \( G \) is Eulerian, then all of its vertices have even order.
   
   Now when we add vertices to \( G \) and \( H \) in order to form homeomorphic graphs, the degrees of the original vertices of \( G \) do not change. Furthermore, since we only add vertices of degree 2, then the resulting graph has only vertices of even degree, and hence this graph is Eulerian also.

   By the same argument, \( H \) must only have vertices of even degree, and \( H \) is therefore Eulerian.

6. No. For example, the graphs below are homeomorphic:

   but only the graph on the right is Hamiltonian.

7. Every complete graph \( K_n \) where \( n > 5 \) has \( K_5 \) as a subgraph.

   \( \therefore \) by the theorem of Kuratowski, \( K_n \) is non-planar for \( n \geq 5 \).

8. a. \( \therefore \) the graph is non-planar.

   b. has subgraph

   which is homeomorphic to

   This is isomorphic to

   which is \( K_{3,3} \).

   \( \Rightarrow \) the graph is non-planar.

   c. can be redrawn as

   which is \( K_{3,3} \).

   \( \therefore \) by the theorem of Kuratowski, the graph is non-planar.